

# Applying inversion to construct rational spiral curves

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## Abstract

A method is proposed to construct spiral curves by inversion of a spiral arc of parabola. The resulting curve is rational of 4-th order. Proper selection of the parabolic arc and parameters of inversion allows to match a wide range of boundary conditions, namely, tangents and curvatures at the endpoints, including those, assuming inflection.

**Keywords:** spiral, transition curve, rational curve, inversion.

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## 1. Introduction

Constructing curves with monotone curvature function (spirals) is a well-known problem in CAD applications. One application of finding spiral arcs with predefined boundary conditions is the design of transition curves, joining two given curves in  $G^2$ -continuous manner.

Another application was proposed in [1]. For a spiral, represented by a set of interpolation nodes with boundary tangents, a region, covering all possible instances of spiral splines, was constructed. The width of the region allows to estimate the determinism of a curve by the given point set, and allows to accord curve design with the requirements, imposed by manufacturing tolerances.

The recent approaches to the problem include approximate solutions for small arcs, solutions for specific boundary conditions, attempts to select spiral segments within traditional polynomial or rational curves. An exact solution was given in [2] in terms of Cornu spiral, extended by two circular arcs.

The author's contribution to the subject is the study of spirals as a special class of planar curves [3, 4, 5]. The main results of this study are the necessary and sufficient conditions of existence of a spiral with given curvature elements at the endpoints. They expand some statements of W. Vogt and A. Ostrowski, well known from [6], onto non-convex spiral arcs, i.e. those with inflection or multiple winding. The requirement of curvature continuity was also removed, which is the case of circular splines with piecewise constant curvature.

Based on this study, a construction in terms of rational curves was developed and is being presented in this note. The construction is straightforward and does not require any heuristic

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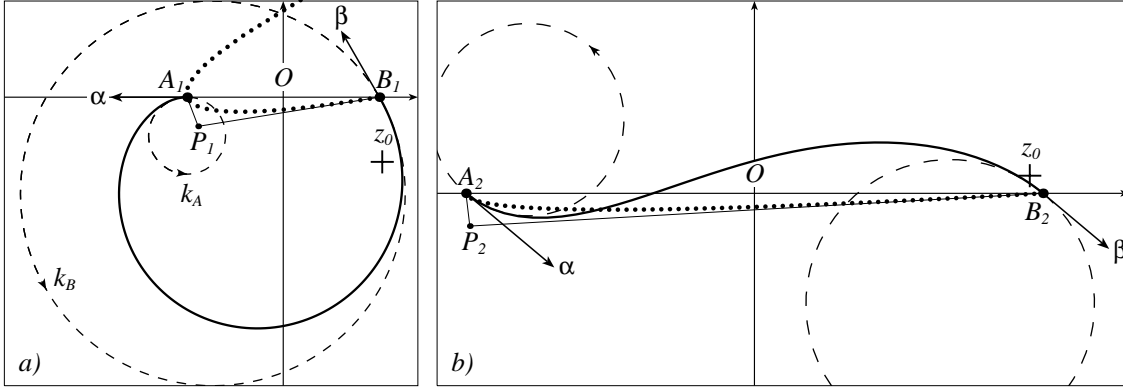


Figure 1.

optimizational or fitting procedures. A wide range of boundary conditions is covered. A possible development of the method to satisfy any boundary conditions, compatible with spirality, is discussed in the last section.

### 1.1. Overview of the method

The proposed method is illustrated by Fig. 1. Suppose, we have to construct a spiral arc from the point  $A_1 = (-1, 0)$  to  $B_1 = (1, 0)$ . Required parameters are shown as tangent vectors and the circles of curvature (dashed) at the startpoint  $A_1$ , marked by  $\alpha$  and  $k_A$ , and, at the endpoint  $B_1$ , by  $\beta$  and  $k_B$ .

First, we construct a specially calculated parabolic arc, shown by dotted line, inscribed into its control polygon  $A_1P_1B_1$ . This is a second order Bezièr curve

$$(x(t), y(t)) = A_1(1-t)^2 + 2P_1(1-t)t + B_1t^2, \quad 0 \leq t \leq 1, \quad (1)$$

with the control point  $P_1 = (p, q)$ . At first glance, this arc has nothing in common with the required curve. The only visible feature is that it is a spiral: the vertex of parabola, although close to the point  $A_1$ , is outside the arc  $A_1B_1$ ,  $t_{vx} \notin (0, 1)$ .

Second, we apply linear fractional transformation (also known as Möbius transformation)

$$W(z; z_0) = \frac{z + z_0}{1 + z_0 z}, \quad z_0 = x_0 + iy_0 \neq \pm 1, \quad (2)$$

where  $z = x(t) + iy(t)$  is a point on the parabola,  $z_0$  is a complex constant. The image of the parabolic arc is the sought for curve. In this example we have calculated  $p \simeq -0.8845$ ,  $q \simeq -0.3033$ ,  $x_0 \simeq 1.0296$ ,  $y_0 \simeq -0.6727$  for prescribed values  $\alpha = -180^\circ$ ,  $k_A = 2.5$ ,  $\beta = 120^\circ$  and  $k_B = 0.5$ .

The solution is based on the following features:

- we can vary 4 free parameters,  $\{p, q, x_0, y_0\}$ , to satisfy 4 required values  $\{\alpha, \beta, k_A, k_B\}$ ;
- Möbius transformation preserves monotonicity of curvature; if  $(p, q)$  defines a spiral parabolic arc (without the vertex inside), the transformed curve is a spiral;

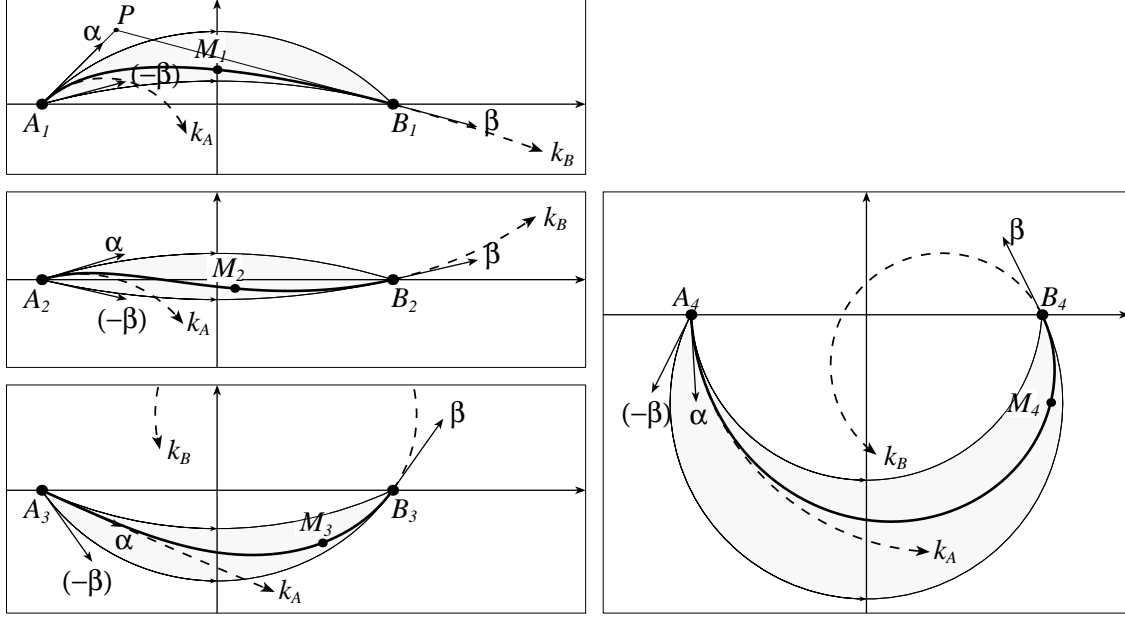


Figure 2.

- original curve (1) being 2nd order polynomial, the image is 4th order rational.

It is well known [7] that transformation (2) includes movements, homothety, symmetry and inversion. The latter provides the necessary flexibility in modifying the form of a curve. In Fig. 1b a curve with inflection was requested ( $\alpha = \beta = -40^\circ$ ,  $3 = k_A > 0 > k_B = -2$ ) and constructed by similar deformation of the parabola with the control point  $P_2$ .

## 2. Definitions and notation

An arc of a curve is described by the functions of the parameter  $t$ :

$$x(t), \quad y(t), \quad z(t) = x(t) + iy(t), \quad \tau(t), \quad k(t), \quad 0 \leq t \leq 1,$$

$\tau$  and  $k$  being the angle of the tangent vector and the curvature at the point  $(x, y)$ . In this article we deal with curves whose function  $k(t)$  is monotone on the segment  $t \in [0, 1]$ , i.e. with spiral arcs.

**Definition 1.** A spiral arc  $\widehat{AB}$  is *short*, if its tangent vector never achieves the direction  $\overrightarrow{BA}$ , opposite to the direction of its chord, except, possibly, at the endpoints.

Another definition, namely, that “a spiral arc is short, if it does not intersect the complement of the chord to the infinite straight line (possibly intersecting the chord itself)”, is equivalent to Def. 1 [4, lemma 4]. Four spiral arcs  $A_i M_i B_i$ ,  $i = 1, 2, 3, 4$ , in Fig. 2 are short. In this article we consider short spirals only, which is sufficient for most of CAD applications. Examples of long ones are:

We denote below a circle of curvature as a quadruple of coordinates of a fixed point, tangent and signed curvature at this point:

$$\mathcal{K}_i = K(x_i, y_i, \tau_i, k_i).$$

It may be a straight line, if  $k_i = 0$ .

To consider properties of the arc  $\widehat{AB}$  with respect to the chord  $\overrightarrow{AB}$  of the length  $|AB| = 2c$ , we choose the coordinate system such that the chord becomes the segment  $[-c, c]$  of the  $x$ -axis. With  $\alpha = \tau(0)$ ,  $k_A = k(0)$ , and  $\beta = \tau(1)$ ,  $k_B = k(1)$ , the boundary circles of curvature take form

$$\mathcal{K}_1 = K(-c, 0, \alpha, k_A), \quad \mathcal{K}_2 = K(c, 0, \beta, k_B). \quad (3)$$

It is convenient to assume homothety with the coefficient  $c^{-1}$ , and to operate on the segment  $[-1, 1]$  of the  $x$ -axis. The coordinates  $x, y$ , arc length  $s$ , and curvature  $k$  become normalized dimensionless quantities, corresponding to  $x/c$ ,  $y/c$ ,  $s/c$  and  $kc$ . With such homothety applied, boundary circles (3) appear as (4).

**Definition 2.** A spiral arc with boundary curvature elements

$$\mathcal{K}_1 = K(-1, 0, \alpha, a), \quad \mathcal{K}_2 = K(1, 0, \beta, b), \quad a = c k_A, \quad b = c k_B. \quad (4)$$

is said to be in *normalized* position. The product  $kc$ , invariant under homotheties, will be referred to as *normalized curvature*.

For each curve  $A_i M_i B_i$  in Fig. 2 two circular arcs are traced from  $A_i$  to  $B_i$ . They share tangent with the spiral at one of the endpoints. These two arcs form a *lense*, enclosing the spiral. The angular width of the lense (signed) is

$$\sigma = \alpha + \beta.$$

Four points  $M_i$  in Fig. 2, four arcs  $A_i M_i B_i$  (the first one is parabola) and four lenses are images of each other under transformation (2). The angle  $\sigma$  remains constant for the four lenses.

### 3. Theoretical background

The following is valid for spirals:

- (i) Monotonicity of curvature is preserved under inversion, increasing curvature being transformed into decreasing, and vice versa [3, theorem 1].
- (ii) The necessary and sufficient condition for the existence of a non-biarc spiral with boundary conditions (3) is inequality

$$Q(\mathcal{K}_1, \mathcal{K}_2) = (k_A c + \sin \alpha)(k_B c - \sin \beta) + \sin^2 \frac{\alpha + \beta}{2} < 0 \quad (5)$$

[4, th. 2]. If  $Q = 0$ , the biarc is the unique spiral, matching these boundary parameters.

(iii) For the existence of a short spiral, it is required additionally that

$$\text{sign}(\alpha + \beta) = \text{sign}(k_B - k_A) \neq 0,$$

[5, th. 1]<sup>1</sup>, or, in details,

$$\begin{aligned} \text{if } k_A < k_B : & \quad \alpha + \beta > 0, & -\pi < \alpha, \beta \leq \pi; \\ \text{if } k_A > k_B : & \quad \alpha + \beta < 0, & -\pi \leq \alpha, \beta < \pi. \end{aligned} \tag{6}$$

(iv) Both  $Q$  and  $\sigma = \alpha + \beta$  are invariant under Möbius transformations.

(v) Short spiral is enclosed in its lense [5, theorem 2].

Below we provide some comments to these statements.

Alternative proof of (i) could be easily derived from the well known facts that (a) the curve in the vicinity of some point intersects its circle of curvature if the curvature is monotone, and (b) the curve is located from one side of this circle, if the point corresponds to vertex. These local properties are evidently invariant under inversions, and no new vertices can appear on the transformed curve.

In (ii) and (5)  $Q = \sin^2 \frac{\psi}{2}$ , where  $\psi$  is the intersection angle of the two circles of curvature.  $Q < 0$  means that  $\psi$  is pure imaginary.  $Q < 0$  can be interpreted as follows: if two given circles are inverted into a concentric pair, taking into account their orientation, the resulting circles will be parallel (and not *anti*parallel, as it will be the case if  $Q > 1$ ).  $Q = 0$  involves  $\psi = 0$ , and corresponds to tangency of two circles (biarc curve can be constructed).  $0 < Q < 1$  means their real intersection, which is not compatible with spirality. Neither is  $Q \geq 1$ .

Statement (iii) is illustrated by Fig. 1 for decreasing curvature ( $\sigma = \alpha + \beta < 0$ ). Note that if we bring the function  $\tau(t)$  to the range  $[-\pi, \pi]$ , then, to preserve continuity, the value  $-\pi$ , not  $+\pi$ , should be assigned to the angle  $\alpha$  of the spiral  $A_1B_1$  in Fig. 1a. This accords to (6). Curvatures  $k(t)$  in Fig. 2 are increasing, and  $\sigma > 0$ .

For (iv) we note that oriented angles  $\psi$  and  $\sigma$  only change sign under inversions, wherefrom the invariance of  $Q, |\sigma|$  follows. Transformation (2) is either identity ( $z_0 = 0$ ), or includes both inversion and symmetry. The type of monotonicity of curvature (increasing/decreasing) is swapped under inversion and restored under symmetry. So does sign  $\sigma$ .

Statement (v) was proven for convex spirals in [8] and for any short spiral in [5]. Fig. 2 demonstrates it, including non-convex case  $A_2M_2B_2$ .

Fig. 3 illustrates inequality  $Q < 0$  (5), for fixed angles  $\alpha, \beta$  taking form

$$Q(a, b; \alpha, \beta) = (a + \sin \alpha)(b - \sin \beta) + \sin^2 \frac{\alpha + \beta}{2} < 0.$$

Angles  $\alpha = 45^\circ, \beta = -15^\circ$  are those of the arc  $A_1M_1B_1$ , the point  $K = (a, b) \simeq (-1.98, -0.10)$  corresponds to normalized boundary curvatures of this arc. The equation  $Q(a, b) = 0$  describes a hyperbola in the plane  $(a, b)$ . Two convex regions  $Q \leq 0$ , bounded by two branches

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<sup>1</sup>Articles [4, 5] are not yet translated in English. The interested reader may find English version of all these facts and proofs in arXiv:math/0601440v2 (*A. Kurnosenko. Around Vogt's theorem*).

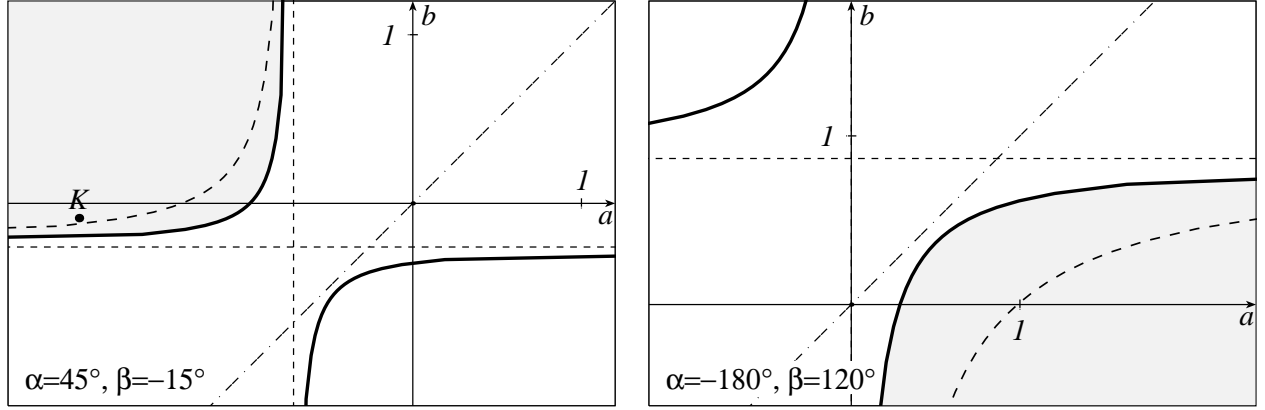


Figure 3.

of the hyperbola, cover the values of boundary curvatures, allowing spiral arcs to be constructed. The upper left branch, lying in the halfplane  $a < b$ , corresponds to increasing curvatures. As the angles in the first example are such that  $\alpha + \beta = 30^\circ > 0$ , corresponding short spirals are of increasing curvature (6); therefore, possible boundary curvatures  $(a, b)$  of short spirals for this case are covered by the upper left region.

The second example in Fig. 3 is drawn for  $\alpha = -180^\circ$ ,  $\beta = 120^\circ$ , the angles of the spiral  $A_1B_1$  in Fig. 1a. As  $\sigma = -60^\circ < 0$ , short curves are of decreasing curvature,  $a > b$ . Possible values of  $(a, b)$  are covered by the lower right hyperbolic region.

The following inequalities just reflect the position of two regions in question with respect to the asymptotae of the hyperbola [4, cor. 2.1]:

$$a \leq b \quad \implies \quad a + \sin \alpha \leq 0, \quad b - \sin \beta \geq 0. \quad (7)$$

Curvatures at the center of the hyperbola ( $a_0 = -\sin \alpha$ ,  $b_0 = \sin \beta$ ) are those of two circular arcs, bounding the lense.

Additional hyperbola, drawn dashed, bounds the region of applicability of the proposed method. This is discussed later (Prop. 4).

#### 4. Transformation of a spiral arc

Now we investigate map (2). This is a particular case of general Möbius transformation  $z \rightarrow \frac{az+b}{cz+d}$ ,  $ad-bc \neq 0$ , keeping points  $(-1, 0)$  and  $(1, 0)$  intact.

**Proposition 1.** *Let two circles of curvature (4) of a spiral arc, and two circles*

$$\mathcal{K}_1^* = K(-1, 0, \alpha^*, a^*), \quad \mathcal{K}_2^* = K(1, 0, \beta^*, b^*),$$

*of another spiral arc are such that*

$$\sigma(\mathcal{K}_1, \mathcal{K}_2) = \sigma(\mathcal{K}_1^*, \mathcal{K}_2^*), \quad Q(\mathcal{K}_1, \mathcal{K}_2) = Q(\mathcal{K}_1^*, \mathcal{K}_2^*). \quad (8)$$

Then transformation (2) with

$$z_0 = \frac{r_0 e^{i\lambda_0} - 1}{r_0 e^{i\lambda_0} + 1}, \quad \text{where} \quad \begin{cases} \lambda_0 = \alpha^* - \alpha = \beta - \beta^*, \\ r_0 = \frac{a + \sin \alpha}{a^* + \sin \alpha^*} = \frac{b^* - \sin \beta^*}{b - \sin \beta} > 0, \end{cases} \quad (9)$$

maps the first pair of circles to the second one.

**Proof.** Conditions (8) can be written as

$$\begin{aligned} \alpha + \beta &= \alpha^* + \beta^*, \\ (a + \sin \alpha)(b - \sin \beta) &= (a^* + \sin \alpha^*)(b^* - \sin \beta^*); \end{aligned} \quad (10)$$

that's why two expressions for  $\lambda_0$  and  $r_0$  in (9) are equivalent. Because  $\text{sign}(\alpha + \beta) = \text{sign}(\alpha^* + \beta^*)$ , two spirals have the same type of monotonicity of curvature. Numerator and denominator in  $r_0$  are non-zero and of equal signs (7). Thus  $r_0 > 0$  in (9) is justified.

Treating  $z$  in  $W(z; z_0)$  as any object, subjected to map (2), we denote images of circles  $\mathcal{K}_{1,2}$  as  $W(\mathcal{K}_{1,2}; z_0)$ . We have to prove that  $W(\mathcal{K}_{1,2}; z_0) = \mathcal{K}_{1,2}^*$ .

Let  $z(s)$  be arc-length parametrization of the circle  $\mathcal{K}_1$ , i.e.

$$z(s) = -1 + \frac{i}{a} e^{i\alpha} (1 - e^{ias}) : \quad z'_s(s) = e^{i(\alpha+as)}, \quad z''_{ss}(s) = i a e^{i(\alpha+as)}.$$

We need not pay special attention to the straight line case  $a = 0$ , because

$$\lim_{a \rightarrow 0} z(s) = -1 + s e^{i\alpha}$$

is the straight line in question. Let  $w(s) = W(z(s); z_0)$  be some parametrization of the image of  $\mathcal{K}_1$  (in general case the parameter  $s$  is not arc length for  $w(s)$ ). Calculating derivatives at  $z(0) = w(0) = -1$  yields:

$$\begin{aligned} w'_s(s) &= z'_s(s) \frac{1 - z_0^2}{[1 + z_0 z(s)]^2}, \\ w'_s(0) &= z'_s(0) \frac{1 - z_0^2}{[1 + z_0 z(0)]^2} = e^{i\alpha} \frac{1 + z_0}{1 - z_0} = e^{i\alpha} \cdot r_0 e^{i\lambda_0} = r_0 e^{i\alpha^*}. \end{aligned}$$

As  $r_0 > 0$ , we get required tangent  $\alpha^*$  for  $w(0)$ . To control curvature, we need the second derivative:

$$\begin{aligned} w''_{ss}(s) &= z''_{ss} \frac{1 - z_0^2}{[1 + z_0 z(s)]^2} - 2z_0 z'_s{}^2 \frac{1 - z_0^2}{[1 + z_0 z(s)]^3}, \\ w''_{ss}(0) &= i a e^{i\alpha} \frac{1 + z_0}{1 - z_0} - 2z_0 e^{2i\alpha} \frac{1 + z_0}{(1 - z_0)^2} = r_0 e^{i\alpha^*} (i a + e^{i\alpha} - r_0 e^{i\alpha^*}). \end{aligned}$$

We separate real and imaginary parts,  $w'_s(0) = u_1 + i v_1$ ,  $w''_{ss}(0) = u_2 + i v_2$ :

$$\begin{aligned} u_1 &= r_0 \cos \alpha^*, & v_1 &= r_0 \sin \alpha^*, \\ u_2 &= r_0 [\cos(\alpha + \alpha^*) - a \sin \alpha^* - r_0 \cos(2\alpha^*)], \\ v_2 &= -r_0 [\sin(\alpha + \alpha^*) + a \cos \alpha^* + r_0 \sin(2\alpha^*)]. \end{aligned}$$

Calculating the curvature of the circle  $w(s)$  at  $s=0$  proves that  $W(\mathcal{K}_1; z_0) = \mathcal{K}_1^*$ :

$$\frac{v_2 u_1 - v_1 u_2}{(u_1^2 + v_1^2)^{3/2}} = \frac{a + \sin \alpha}{r_0} - \sin \alpha^* = a^*.$$

Similar calculations for the second pair of circles can be omitted. The required values  $\beta^*$  and  $b^*$  result from (8) and from the invariance of  $Q, \sigma$  (iv). Namely, denote the image  $W(\mathcal{K}_2; z_0)$  as  $K(1, 0, \bar{\beta}, \bar{b})$ . Then

$$\sigma(\overbrace{W(\mathcal{K}_1; z_0)}^{\mathcal{K}_1^*}, W(\mathcal{K}_2; z_0)) \stackrel{\text{(iv)}}{=} \sigma(\mathcal{K}_1, \mathcal{K}_2) \stackrel{\text{(8)}}{=} \sigma(\mathcal{K}_1^*, \mathcal{K}_2^*).$$

Thus obtained equation  $\sigma(\mathcal{K}_1^*, W(\mathcal{K}_2; z_0)) = \sigma(\mathcal{K}_1^*, \mathcal{K}_2^*)$  is in fact  $\alpha^* + \bar{\beta} = \alpha^* + \beta^*$ , and yields  $\bar{\beta} = \beta^*$ . The same reasoning for  $Q$  yields  $\bar{b} = b^*$ , i.e.  $W(\mathcal{K}_2; z_0) = \mathcal{K}_2^*$ .  $\square$

## 5. Spiral parabolic arc

In this section we explore a spiral parabolic arc from the viewpoint of its boundary conditions and invariants. The equation (1) with the control point  $(p, q)$  can be rewritten as

$$\begin{aligned} x(t) &= -(1-t)^2 + 2p(1-t)t + t^2, \\ y(t) &= 2qt(1-t), \end{aligned} \quad q \neq 0, \quad 0 \leq t \leq 1. \quad (11)$$

Calculate

$$\begin{aligned} x'(t) &= 2(1+p)(1-t) + 2(1-p)t, \quad y'(t) = 2q(1-t) - 2qt, \\ g(t) &= \sqrt{x'^2(t) + y'^2(t)} = 2\sqrt{(1-t)^2 h_1^2 + (1-t)t(h_1^2 + h_2^2 - 4) + t^2 h_2^2}, \end{aligned}$$

where

$$h_1 = |AP| = \sqrt{(1+p)^2 + q^2} \quad \text{and} \quad h_2 = |PB| = \sqrt{(1-p)^2 + q^2}. \quad (12)$$

Since  $\cos \tau(t) = x'(t)/g(t)$ ,  $\sin \tau(t) = y'(t)/g(t)$ , the boundary angles are defined by

$$\cos \alpha = \frac{1+p}{h_1}, \quad \sin \alpha = \frac{q}{h_1}, \quad \cos \beta = \frac{1-p}{h_2}, \quad \sin \beta = \frac{-q}{h_2}, \quad (13)$$

and invariant  $\sigma = \alpha + \beta$  by

$$\cos \sigma = \frac{1 - p^2 + q^2}{h_1 h_2}, \quad \sin \sigma = \frac{-2pq}{h_1 h_2}. \quad (14)$$

Curvatures are

$$k(t) = \frac{y''x' - x''y'}{g^3} = \frac{-8q}{g(t)^3}, \quad a = k(0) = \frac{-q}{h_1^3}, \quad b = k(1) = \frac{-q}{h_2^3}. \quad (15)$$

Invariant (5), expressed as a function of the control point, looks like

$$Q(p, q) = \frac{1}{2} + \frac{p^2 + q^2 - 1}{2h_1 h_2} - \frac{q^2(2p^2 + 2q^2 + 1)}{h_1^3 h_2^3}. \quad (16)$$



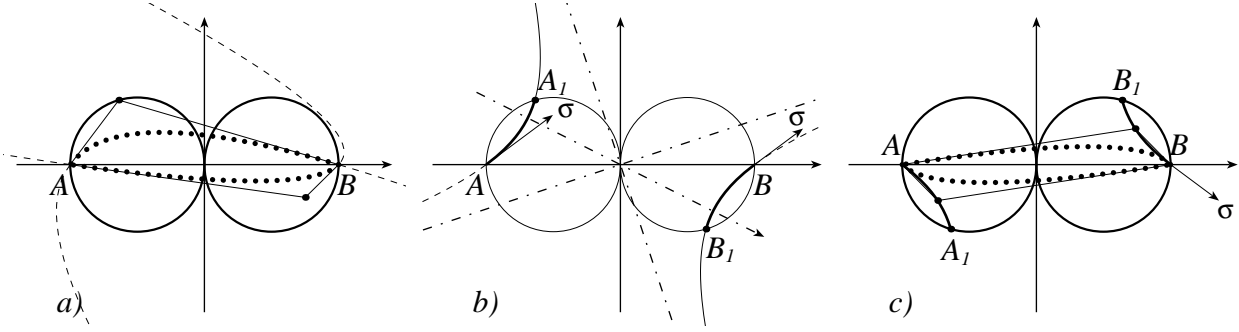


Figure 4.

**Proposition 2.** *Parabolic arc (11) is spiral if and only if the control point  $(p, q)$  satisfies inequalities*

$$\text{where } \begin{aligned} &F_1(p, q) \cdot F_2(p, q) \leq 0, \quad q \neq 0, \\ &F_1(x, y) = x^2 + x + y^2, \quad F_2(x, y) = x^2 - x + y^2. \end{aligned} \quad (17)$$

**Proof.** Differentiating of  $k(t)$  (15) yields

$$\begin{aligned} k'(t) &= \frac{192q[(p^2 + q^2)(2t - 1) - p]}{g(t)^5} = \frac{192q}{g(t)^5} [-(1 - t)F_1(p, q) + tF_2(p, q)], \\ k'(0) &= -\frac{6q}{h_1^5} F_1(p, q), \quad k'(1) = \frac{6q}{h_2^5} F_2(p, q). \end{aligned}$$

To avoid vertex to occur in  $t \in (0, 1)$ , we require that the derivative did not change sign within this interval, allowing the vertex at  $t=0$  or  $t=1$ :  $k'(0)k'(1) \geq 0$ . This yields the first inequality in (17); the second one prevents parabola from degenerating into a straight line.  $\square$

Fig. 4a illustrates (17): the control point should be taken within or on the boundary of any of two circles  $F_{1,2}(x, y) = 0$ , but not on the  $x$ -axis.

**Proposition 3.** *The locus of control points  $(p, q)$ , yielding  $\sigma = \text{const}$ , is*

$$H(x, y) \equiv \sin \sigma (1 - x^2 + y^2) + 2xy \cos \sigma = 0, \quad \text{sign}(xy) = -\text{sign } \sigma. \quad (18)$$

**Proof.** The proof follows immediately from (14).  $\square$

The locus (18) is an equilateral hyperbola, centered at the origin, passing through points  $A$  and  $B$  (Fig. 4b). The second equality in (18) keeps only the part of the hyperbola, lying, if  $\sigma > 0$ , in quadrants II, IV. Finally, subarcs  $AA_1$  and  $B_1B$  provide control points  $(p, q)$ , corresponding to spiral parabolic arcs. For  $\sigma < 0$  we get the picture, symmetric about the  $x$ -axis, with subarcs  $AA_1$  and  $B_1B$  in quadrants I, III (Fig. 4c).

Both conditions (18) in polar coordinates  $(\rho, \xi)$  look like the polar equation of the hyperbola and the intervals for  $\xi$ :

$$\rho(\xi) = \sqrt{\frac{\sin \sigma}{\sin(\sigma - 2\xi)}}, \quad \begin{aligned} \sigma > 0 : \quad & \frac{\sigma}{2} - \frac{\pi}{2} < \xi < 0, \quad \frac{\pi}{2} + \frac{\sigma}{2} < \xi < \pi; \\ \sigma < 0 : \quad & 0 < \xi < \frac{\pi}{2} + \frac{\sigma}{2}, \quad -\pi < \xi < \frac{\sigma}{2} - \frac{\pi}{2}. \end{aligned} \quad (19)$$

**Proposition 4.** *Under conditions*

$$0 < |\sigma_0| < \frac{\pi}{2}, \quad (20a)$$

$$Q_0 \leq Q_{max} = -\frac{w^6(w^2 + 2)}{(1 - w^2)(w^2 + 1)^3}, \quad w = \sqrt[3]{\tan \frac{\sigma_0}{2}}. \quad (20b)$$

there exist two parabolic spiral arcs (11), such that  $Q(p, q) = Q_0$  and  $\sigma(p, q) = \sigma_0$ .

**Proof.** The direction of tangent to the hyperbola  $H(x, y) = 0$  at points  $A$  and  $B$  is  $\sigma_0$  (Fig. 4b). Condition (20a) assures the existence of the non-vanishing arcs  $AA_1$  and  $BB_1$  within limiting circles (17).

Consider the behavior of the invariant  $Q(p, q)$  while the control point moves along the path  $AA_1$  (or  $BB_1$ ).  $Q(p, q)$  can be considered as the function  $Q(\xi)$  of the polar angle. To get it, we first express the product  $h_1 h_2$  from (14):

$$h_1 h_2 = \frac{-2pq}{\sin \sigma_0} = \frac{-2\rho^2(\xi) \sin \xi \cos \xi}{\sin \sigma_0} = \frac{\sin 2\xi}{\sin(2\xi - \sigma_0)}.$$

Substituting  $p^2 + q^2 = \rho^2(\xi)$ ,  $q^2 = \rho^2(\xi) \sin^2 \xi$ , and the above product into (16), we obtain

$$\begin{aligned} Q(\xi; \sigma_0) &= f_2(\sigma_0) - \sin^3 \sigma_0 f_1(\xi), \\ \text{where } f_2(\sigma) &= \cos^3 \sigma - \frac{3}{2} \cos \sigma + \frac{1}{2}, \\ \text{and } f_1(\xi) &= \frac{1 + 4 \cos^2 \xi - 8 \cos^4 \xi}{8 \cos^3 \xi \sin \xi} = \frac{\tan^4 \xi + 6 \tan^2 \xi - 3}{8 \tan \xi}. \end{aligned} \quad (21)$$

$Q(\xi; \sigma_0)$  is a monotone function of  $\xi$ :

$$\frac{dQ}{d\xi} = \frac{-3 \sin^3 \sigma_0}{8 \cos^4 \xi \sin^2 \xi}$$

As  $\xi$  decreases from  $A$  to  $A_1$  and farther to the asymptota of the hyperbola,  $Q$  increases

$$\text{from } \lim_{\xi \rightarrow \pi-0} Q(\xi; \sigma_0) = -\infty \quad \text{to} \quad \lim_{\xi \rightarrow \frac{\pi}{2} + \frac{\sigma_0}{2} + 0} Q(\xi; \sigma_0) = 1,$$

remaining still negative at  $A_1$ , because parabolic arc is still spiral (5). We denote this value as  $Q_{max}$ . To calculate it, find points  $A_1, B_1$  by intersecting the pair of circles  $F_1(p, q)F_2(p, q) = 0$  with the hyperbola, i.e. from the equations

$$(p^2 + q^2)^2 - p^2 = 0, \quad p = \rho(\xi) \cos \xi, \quad q = \rho(\xi) \sin \xi.$$

They are simplified to  $\rho^2(\xi) = \cos^2 \xi$  and, finally, to

$$\tan \xi \left( \tan^3 \xi \tan \frac{\sigma_0}{2} + 3 \tan \xi \tan \frac{\sigma_0}{2} + 1 - \tan^2 \frac{\sigma_0}{2} \right) = 0.$$

We ignore the root  $\tan \xi = 0$  (intersection points  $A, B$ ), and substitute  $\tan \frac{\sigma_0}{2} = w^3$ :

$$\begin{aligned} & w^3 \tan^3 \xi + 3w^3 \tan \xi + 1 - w^6 = 0 \\ \implies & (w \tan \xi + 1 - w^2) [w^2 \tan^2 \xi + w(w^2 - 1) \tan \xi + w^4 + w^2 + 1] = 0. \end{aligned}$$

The unique real root  $\tan \xi_1 = \frac{w^2 - 1}{w}$  gives both points  $A_1$  and  $B_1$ . Calculating  $Q(\xi_1; \sigma_0)$  yields the expression (20b) for  $Q_{max}$ .  $\square$

The plot  $Q_{max}(\sigma)$  is shown in Fig. 5. The region of curvatures, rejected by condition (20b), was shown in Fig. 3 as the band-like zone between the line  $Q(a, b) = 0$  and the dashed line  $Q(a, b) = Q_{max}$ . The narrower is the lense, the narrower is the rejection region  $Q_{max} < Q \leq 0$ . In other words, our method is not applicable, when boundary circles of curvature are close to tangency.

Now we find the explicit solution  $\xi = \xi_0$  for the equation  $Q(\xi; \sigma_0) = Q_0$ , i.e. find the control points on hyperbola (19) with  $\sigma = \sigma_0$ , yielding the parabolic arc with predefined value of invariants  $\sigma_0, Q_0$ .

We denote for brevity

$$Q_1 = \frac{Q_0 - f_2(\sigma_0)}{\sin^3 \sigma_0}, \quad m = \sqrt[3]{1 + Q_1^2}, \quad n = \sqrt{m^2 + m + 1}, \quad (22)$$

and

$$\begin{aligned} r_1 &= \sqrt{m - 1} = \frac{|Q_1|}{n}, \\ r_2 &= \sqrt{2n - (m + 2)} = \frac{m\sqrt{3}}{\sqrt{2n + m + 2}}, \\ r_{12} &= r_2^2 - r_1^2 = 2n - (2m + 1) = \frac{3}{2n + 2m + 1}. \end{aligned}$$

Each of the above definitions for  $r$ 's is represented in the alternative form, intended to avoid loss of precision while subtracting close positive numbers in calculations. For the same reason, the expression for  $\theta_0$  is splitted in (23), and  $r_2 - r_1$  is replaced by  $\frac{r_{12}}{r_1 + r_2}$ .

**Proposition 5.** *Solutions  $\xi_0, \bar{\xi}_0$  of the equation  $Q(\xi; \sigma_0) = Q_0$ , and corresponding control points  $(p_{1,2}, q_{1,2})$  are given by*

$$\begin{aligned} \theta_0 &= -r_1 \operatorname{sign} Q_1 - r_2 \operatorname{sign} \sigma_0 = \begin{cases} -\frac{r_{12}}{r_1 + r_2} \operatorname{sign} \sigma_0, & \text{if } \sigma_0 Q_1 < 0; \\ -(r_1 + r_2) \operatorname{sign} \sigma_0, & \text{if } \sigma_0 Q_1 \geq 0; \end{cases} \\ \xi_0 &= \arctan \theta_0, \quad (p_1, q_1) = (\rho(\xi_0) \cos \xi_0, \rho(\xi_0) \sin \xi_0), \\ \bar{\xi}_0 &= \xi_0 + \pi, \quad (p_2, q_2) = (-p_1, -q_1). \end{aligned} \quad (23)$$

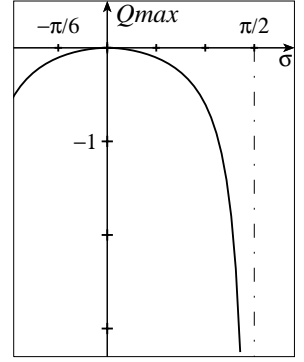


Figure 5.

**Proof.** Equation (21), namely  $Q_0 = f_2(\sigma_0) - \sin^3 \sigma_0 f_1(\xi)$ , can be transformed to

$$\theta^4 + 6\theta^2 + 8Q_1\theta - 3 = 0, \quad \text{with } \theta = \tan \xi, \quad (24)$$

and  $Q_1$ , defined by (22). Following Descartes–Euler’s method, find the cubic resolvent of (24) and its roots:

$$(z + 4)^3 - 64(1 + Q_1^2) = 0, \quad \begin{aligned} z_1 &= -4 + 4m, \\ z_{2,3} &= -4 + 4me^{\pm \frac{2}{3}i\pi}. \end{aligned}$$

Now one should choose signs for  $\zeta_i = \pm\sqrt{z_i}$  to satisfy  $\zeta_1\zeta_2\zeta_3 = -8Q_1$ . We get it, assuming

$$\begin{aligned} \zeta_1 &= -\text{sign } Q_1 \sqrt{z_1} = -2r_1 \text{sign } Q_1, \\ \zeta_{2,3} &= +\sqrt{z_{2,3}} = r_2 \pm i\sqrt{2n + m + 2}, \end{aligned}$$

thus obtaining  $\zeta_1\zeta_2\zeta_3 = -8 \text{sign } Q_1 \sqrt{m^3 - 1} = -8 \text{sign } Q_1 \cdot |Q_1| = -8Q_1$ . So, both real roots of (24) are equal to

$$\theta_{1,2} = \frac{1}{2}\zeta_1 \pm \frac{1}{2}(\zeta_2 + \zeta_3) = -r_1 \text{sign } Q_1 \pm r_2.$$

and are of opposite signs:

$$\theta_1\theta_2 = r_1^2 - r_2^2 = -r_{12} < 0.$$

From (14) we note that

$$\sin \sigma_0 = -\frac{2\rho^2(\xi) \sin \xi \cos \xi}{h_1 h_2} \implies \text{sign } \sigma_0 = -\text{sign } \tan \xi = -\text{sign } \theta.$$

So, the only admissible root  $\theta_0$  from  $\theta_{1,2}$  is of the sign, opposite to that of  $\sigma_0$ :

$$\begin{aligned} \text{if } \sigma_0 > 0, Q_1 < 0: \quad \theta_0 &= r_1 - r_2 < 0; \\ \text{if } \sigma_0 > 0, Q_1 \geq 0: \quad \theta_0 &= -r_1 - r_2 < 0; \\ \text{if } \sigma_0 < 0, Q_1 \leq 0: \quad \theta_0 &= r_1 + r_2 > 0; \\ \text{if } \sigma_0 < 0, Q_1 > 0: \quad \theta_0 &= -r_1 + r_2 > 0. \end{aligned}$$

This selection is unified in (23). □

Thus found  $\xi = \arctan \theta_0$  provides two symmetric parabolas, shown in Fig. 4c.

## 6. Algorithm

Now we summarize step by step the construction of a short spiral arc with predefined boundary conditions.

**Step 1.** Transformation to the chord’s coordinate system.

Given boundary conditions  $\mathcal{K}_M = K(x_1, y_1, \tau_1, k_1)$  at the startpoint  $M$  (Fig. 6a), and  $\mathcal{K}_N = K(x_2, y_2, \tau_2, k_2)$  at the endpoint  $N$ , transform them by orthogonal transformation to

$$\mathcal{K}_1^* = K(-1, 0, \alpha^*, a^*), \quad \mathcal{K}_2^* = K(1, 0, \beta^*, b^*),$$

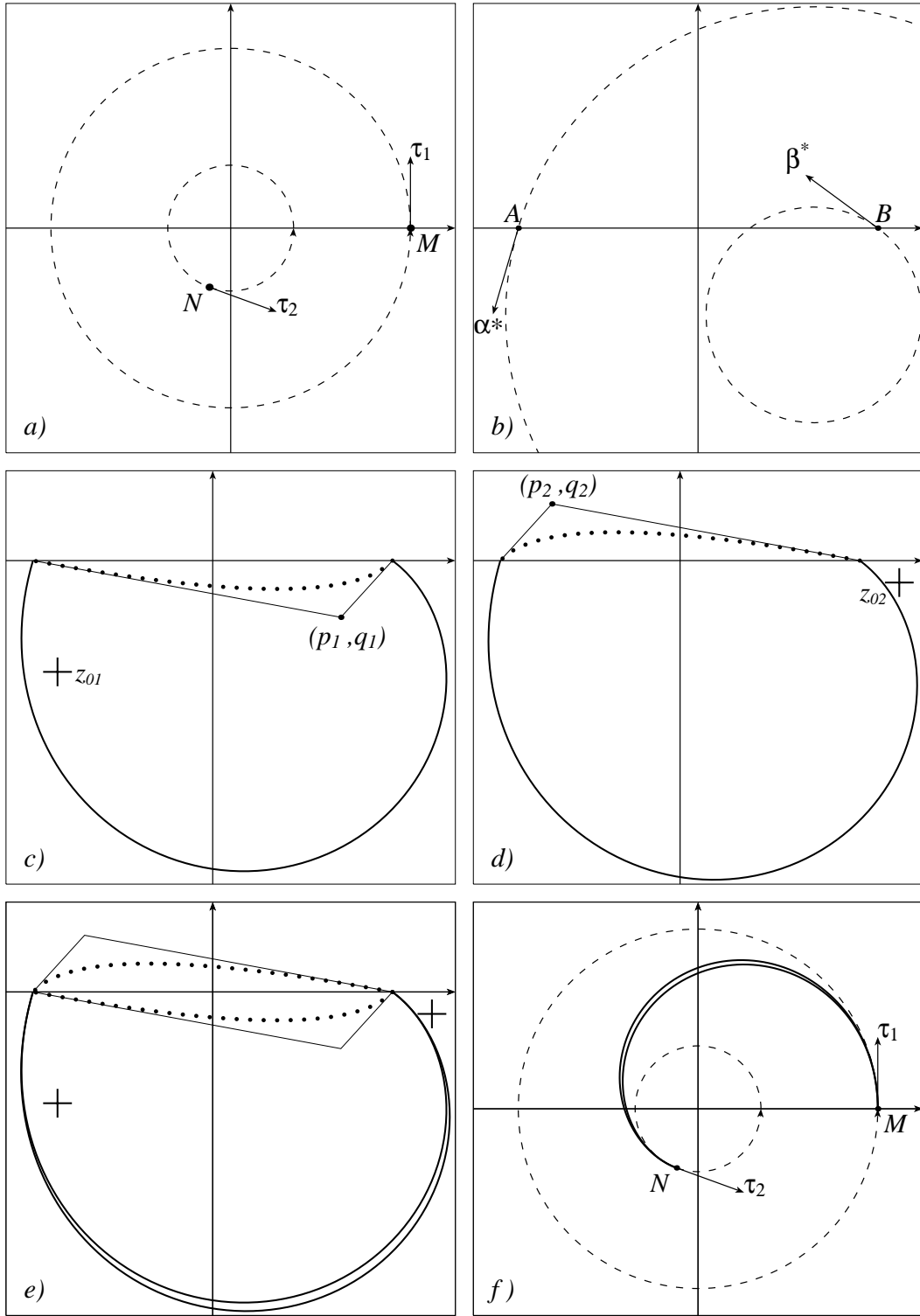


Figure 6.

where

$$\begin{aligned} \alpha^* &= \tau_1 - \mu, & a^* &= k_1 c & c &= \frac{1}{2} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \\ \beta^* &= \tau_2 - \mu, & b^* &= k_2 c. & \mu &= \arg[(x_2 - x_1) + i(y_2 - y_1)], \end{aligned}$$

The transformed configuration is shown in Fig. 6b.

**Step 2.** Check solvability of the problem.

- Calculate  $Q_0 = Q(\mathcal{K}_1^*, \mathcal{K}_2^*)$  and check inequality (5). If  $Q_0 > 0$ , boundary conditions are invalid, irrespective of the proposed method: no such spiral exists. If  $Q_0 = 0$ , biarc is unique spiral. Continue, if  $Q_0 < 0$ .
- As specified in (6), bring the boundary angles to the halfintervals  $(-\pi, \pi]$  or  $[-\pi, \pi)$ . Namely, if  $|\alpha^*| = \pi$  or  $|\beta^*| = \pi$ , replace  $+\pi$  by  $-\pi$  for the case of decreasing curvature. Calculate  $\sigma_0 = \alpha^* + \beta^*$ , check condition (6). If it fails, no short spiral exists.
- Verify conditions (20). If  $Q_0 > Q_{max}$  or  $|\sigma_0| \geq \pi/2$ , the proposed algorithm is not applicable.

**Step 3.** Find spiral parabolic arc (11) such that its boundary tangents  $\alpha, \beta$  and curvatures  $a, b$  satisfy equations (10). Two solutions,  $(p_{1,2}, q_{1,2})$ , are explicitly given above (23). The explicit expression for  $Q_1$  in (22) is

$$Q_1 = \cot \sigma_0 + \frac{(a^* + \sin \alpha^*)(b^* - \sin \beta^*)}{\sin^3 \sigma_0}.$$

Two parabolic arcs are shown in Fig. 6c,d by dotted lines.

**Step 4.** Apply this step to each of two just constructed parabolic arcs.

Calculate boundary tangents  $\alpha, \beta$  from (13) and curvatures  $a, b$  from (15). Define  $r_0, \lambda_0$  from (9). Transforming the parabolic arcs by (2), get the sought for rational curve:

$$\begin{aligned} X(t) + iY(t) &= \frac{x_0(x^2 + y^2 + 1) + x(1 + x_0^2 + y_0^2) + i[y(1 - x_0^2 - y_0^2) + y_0(1 - x^2 - y^2)]}{1 + 2xx_0 - 2yy_0 + (x^2 + y^2)(x_0^2 + y_0^2)} = \\ &= \frac{r_0^2 l_1^2 - l_2^2 + 2ir_0[2y \cos \lambda_0 - (x^2 + y^2 - 1) \sin \lambda_0]}{r_0^2 l_1^2 - 2r_0[2y \sin \lambda_0 + (x^2 + y^2 - 2) \cos \lambda_0] + l_2^2}, \end{aligned}$$

where  $x(t), y(t)$  are abbreviated to  $x, y$ , and  $l_1^2 = (x+1)^2 + y^2$ ,  $l_2^2 = (x-1)^2 + y^2$ . The second expression is tolerant to the case  $z_0 = \infty$ , which is described by finite values  $r_0, \lambda_0$ .

Two solutions are shown combined in Fig. 6e.

**Step 5, final.** Return to the original coordinate system (Fig. 6f).

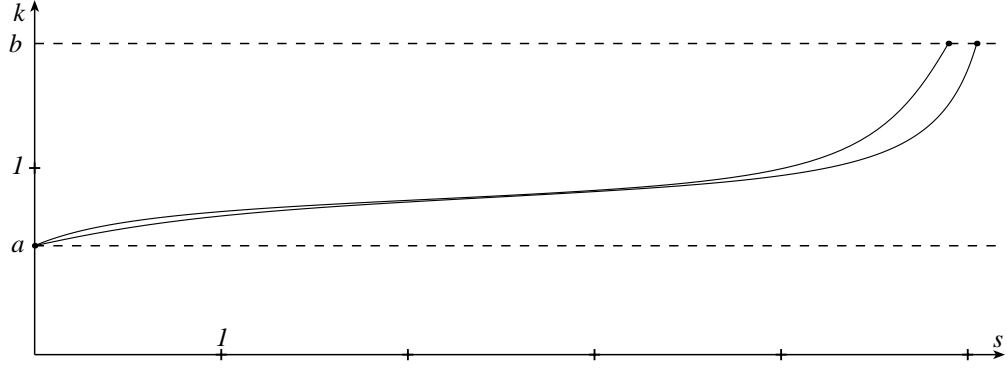


Figure 7.

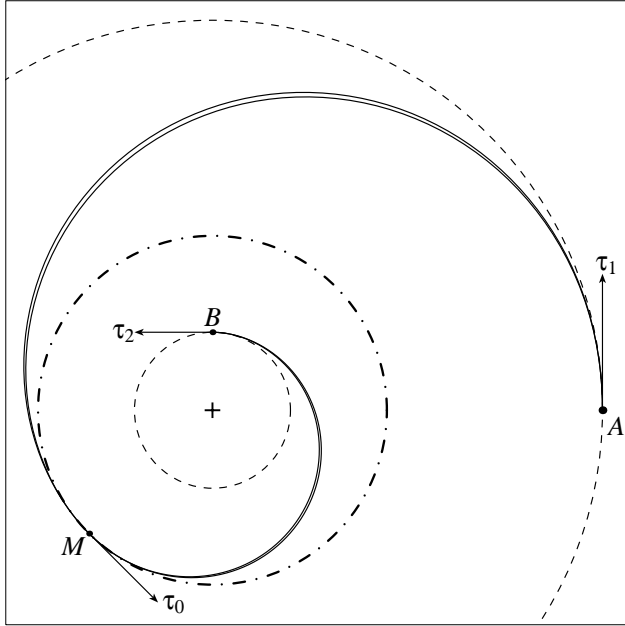


Figure 8.

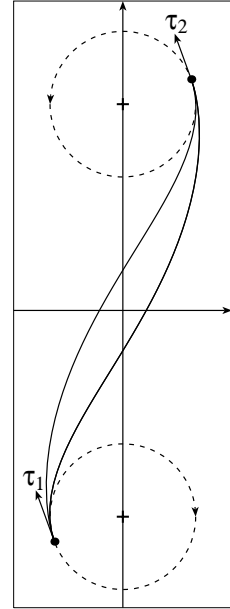


Figure 9.

## 7. Illustrations

In Fig. 6 transition curves, joining two concentric circles, have been constructed. Fig. 7 adds the curvature plots with respect to arc length for both solutions.

In Fig. 8 the boundary circles are also concentric, but the tangents at points  $A$  and  $B$  are such that no short joining spiral exist: condition (6) is violated. To construct a long one, we introduce the point  $M$  at the polar halfway from  $A$  to  $B$ , and new intermediate circle of curvature, concentric to both given ones. Two shorts spirals are constructed on two chords,  $AM$  and  $MB$ . Combining them together, we obtain four variants of the long spiral  $AMB$ ,

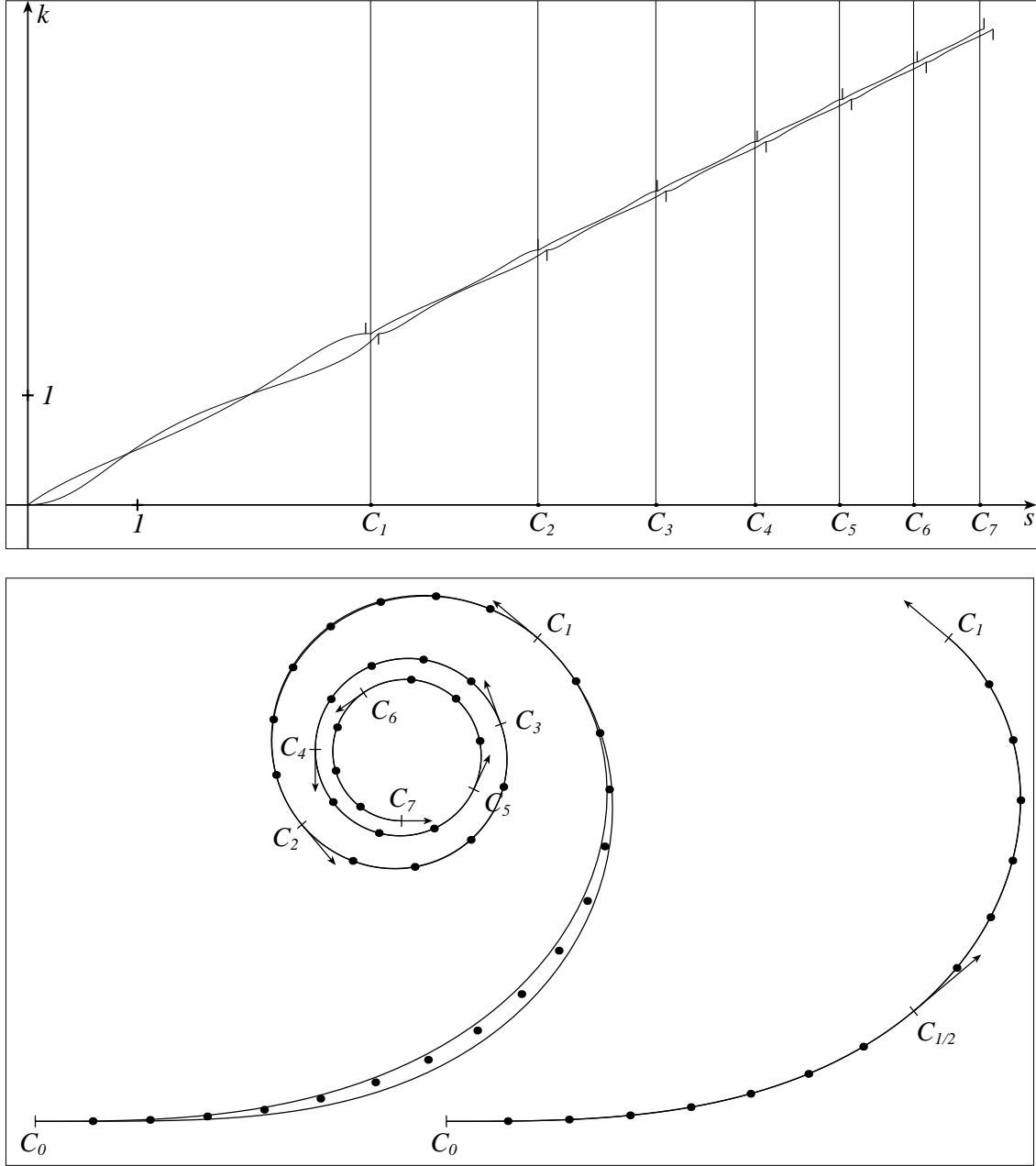


Figure 10.

and the curvature remains continuous at  $M$ . In particular, taking radius  $R_M = \sqrt{R_A R_B}$ , we get similarity of boundary conditions. Arcs  $MB$  can be obtained just by rotation and homothety, applied to both arcs  $AM$ .

In Fig. 9 symmetric boundary conditions are given: tangents are parallel, and  $k_A < 0 < k_B = |k_A|$ . Curvature changes sign, inflection point on the transition curve is therefore required. It does appear in both solutions.



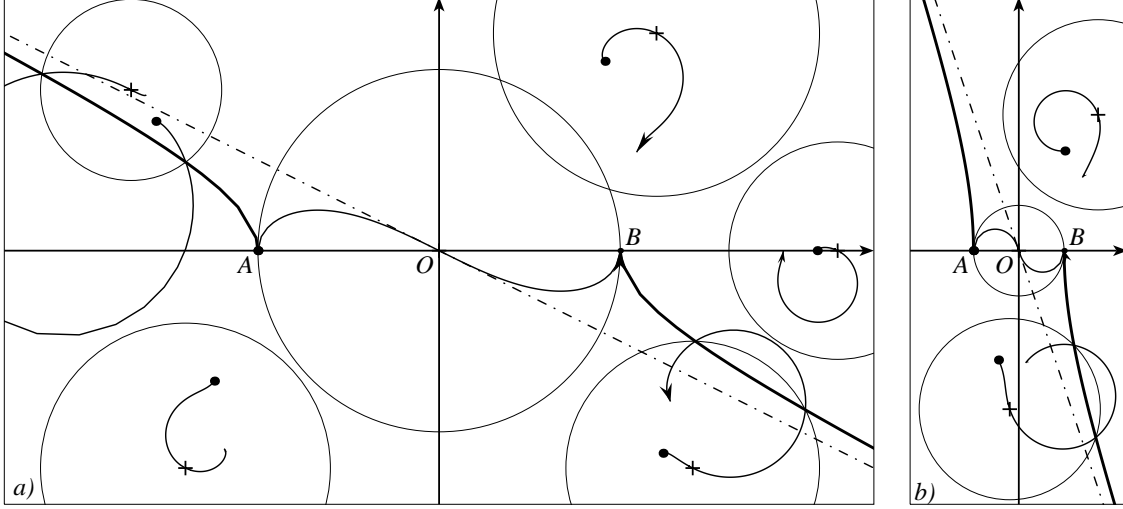


Figure 11.

One could expect a symmetric solution, but, instead, we get the symmetry among two solutions. The problem could be resolved, as in the previous example, by introducing the intermediate inflection point at the coordinate origin.

In Fig. 10 the long arc  $C_0C_7$  of Cornu spiral is drawn by dots and subdivided into 7 short subarcs,  $C_iC_{i+1}$ . The boundary conditions  $(x_i, y_i, \tau_i, k_i)$  are borrowed from those of Cornu spiral at points  $C_i$ . In this example every point  $C_i$  is taken almost as far from  $C_{i-1}$  as limitation (20b) allows. As before, we get two solutions on every segment  $C_iC_{i+1}$ . Differences between two solutions and Cornu spiral itself are well visible on the segment  $C_0C_1$ . This can be avoided, as shown on the right side, by inserting an intermediate point  $C_{1/2}$  between  $C_0$  and  $C_1$ .

The curvature plots with respect to arc length for approximations of Cornu spiral are also shown in Fig. 10 for both solutions on every segment of the curve  $C_0C_7$ . Abscissas of vertical lines correspond to arc length of Cornu spiral in points  $C_i$ . Small vertical marks separate arcs of approximating curves. Proportionality  $k(s) \sim s$  of Cornu spiral is well reproduced in approximations.

## 8. Conclusions

Note that we get much more flexibility in form control by involving spiral arcs of other conics or other spirals: basic Proposition 1 is independent of the kind of spiral involved. Arcs within a quarter of an ellipse are spirals, and, subjected to transformation (2), provide another variety of spiral arcs. Being 2-nd order rationals, they also yield 4-th order rational spirals. So do hyperbolic arcs.

Fig. 11 shows inversions of a half-hyperbola, traced from the vertex at point A through infinity to the second vertex at B. Several inversions of this branch of hyperbola are shown, every example with the circle of inversion. Inversion with respect to the unit circle produces

a curve  $AOB$ , known as hyperbolic lemniscate. Although the original curve has discontinuity at infinity, this feature disappears under inversion. The infinite point goes to the center of inversion and becomes an ordinary, infinitely differentiable, point of the curve-image.

Varying eccentricity of conic gives additional possibilities in constructing rational spirals. A *family* of curves is expected as a solution. Starting with hyperbola, one could get a desired symmetric solution for the symmetric problem in Fig. 9.

Preliminary investigation shows that the region of boundary conditions (20) can be extended to  $|\sigma_0| \leq \pi$  and  $Q_0 < 0$ . Boundary angles of the lemniscate  $AOB$  are  $\alpha = \beta = \pi/2$ , its lense is the unit circle, and the width of the lense is  $\sigma = \pi$ . Taking eccentricity  $e \gg 1$ , two branches can be made anyhow close to parallel lines (Fig. 11*b*). The image can be anyhow close to a biarc curve, and  $Q$  close to 0.

In this article we have considered parabolic arcs only, thus illustrating the simplest, but sufficiently powerful, version of the proposed method.

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